On character varieties of 3-manifold groups

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A character-buildier

- A dialog of a Geometric Topologist (GT) with an Algebraic Geometer (AG)
- GT: I would like to discuss with you character varieties of finitely-presented groups, but ...
- AG: Very commendable, everybody should study varieties!
- GT: But, let us agree, not to interrupt and not to insult each other!
- AG: I will try my best.
- GT: ... but, my knowledge of algebraic geometry is very limited at best.
- AG: Lazy ignoramus!
- GT: As I said, let us not insult each other!

What do we care about

- GT: The objects I care about are *representation varieties* $Hom(\pi, G)$, $G = SL(2, \mathbf{C}), SU(2)$, π is a finitely-presented group...
- ... and their quotient-spaces such as $Hom(\pi, G)/G$, where G acts on $Hom(\pi, G)$ by composition with inner automorphisms of G.
- AG (interrupting): OK, so your $Hom(\pi, G)$ and $Hom(\pi, G)/G$ are sets!
- GT: That much any lazy ignoramus knows, but I want more than that, I would like these sets to be manifolds.
- AG: Then you are out of luck. Do you like non-Hausdorff manifolds?
- GT: No, I am a geometric topologist, not a general topologist.
- AG: Not a problem. $Hom(\pi, SU(2))/SU(2)$ is Hausdorff (I am assuming you mean "with classical topology", since you dislike non-Hausdorff spaces), as for $Hom(\pi, SL(2, \mathbf{C}))/SL(2, \mathbf{C})$, you just have to use its Hausdorffication.

Digression: A non-Hausdorff example

- Take $\pi \cong \mathbb{Z}$ and consider $\rho : \pi \to G = SL(2, \mathbf{C})$ sending the generator 1 to the matrix
- $$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
- Next, take a matrix
$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \lambda < 1$$
- and consider the sequence of conjugates
$$A_n = B^n A B^{-n} = \begin{bmatrix} 1 & 2\lambda^n \\ 0 & 1 \end{bmatrix}$$
and corresponding conjugate representations $\rho_n : 1 \to A_n$.
- In the limit
$$\lim_{n\to\infty} A_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
which is not conjugate to A, of course.

Digression: A non-Hausdorff example

- Therefore, the projection of ρ to $Hom(\pi, G)/G$ cannot be separated from the equivalence class of the trivial representation ρ_o.
- Note that in this example, the orbits $G \cdot \rho$ and $G \cdot \rho_o$ are distinct but their closures intersect.

What kind of a set is it?

- Instead of the G-orbit equivalence relation you should use the
- extended orbit-equivalence relation generated by

$$\rho_1 \sim \rho_2 \iff \overline{G \cdot \rho_1} \cap \overline{G \cdot \rho_2} \neq \emptyset.$$

- Luckily for you, since $G = SL(2, \mathbf{C})$, this equivalence relation is the same as the orbit equivalence unless representations are conjugate to upper-triangular ones.
- This quotient is denoted: $X(\pi, G) = Hom(\pi, G)//G$.
- **Note.** The same quotient construction works for other *reductive group actions* on affine complex-algebraic varieties, not necessarily representation varieties. This is one of the key results of GIT, *Geometric Invariant Theory*. Good references are [D] and [N].

Is it a variety?

- GT: Wonderful! Finally, it feels like we are speaking the same language.
- AG: Not so fast. The best way to describe this quotient is as $Spec(R^G_{Hom(\pi,G)})$, with $R = R_{Hom(\pi,G)}$ the coordinate ring of the algebraic set $Hom(\pi, G) \subset \mathbf{C}^N$:

$$R = \mathbf{C}[x_1, ..., x_N]/\sqrt{I_{Hom(\pi,G)}},$$

  $I_{Hom(\pi,G)}$ is the ideal of $Hom(\pi, G)$.
- Lastly, $R^G \subset R$ is the subring of G-invariants.

A commutative algebra digression

- **Note.** Here is what AG means: The trick is to think not in terms of (algebraic) sets but (polynomial) functions on these sets. Polynomial functions on an algebraic subset $V \subset \mathbf{C}^N$ are the restrictions of polynomial functions on $\mathbf{C}^n$. The kernel of this restriction map is the ideal I_V of polynomials vanishing on V. Then $R = \mathbf{C}[x_1, ..., x_N]/I$ is the *coordinate ring* of the variety V (the ring of polynomial functions on V). The ring of functions on a quotient of V by G should be the ring R^G of functions on V invariant under G; we just have to find an (algebraic set) whose ring of functions is R^G. Since $\mathbf{C}[x_1, ..., x_N]$ is Noetherian, its quotient R is also Noetherian and, thus, the subring R^G as well (quotients and subs of Noetherian rings are Noetherian). Thus, R^G is finitely generated, hence, isomorphic to a ring $\mathbf{C}[y_1, ..., y_M]/J$. We get the variety $W = \{y : g(y) = 0, \forall g \in J\} \subset \mathbf{C}^M$, the coordinate ring of this variety is R^G. Declare W to be the quotient $V//G$.

Is it a variety?

- GT: Hmm. Whatever. I liked the other description much better. Is this quotient a manifold now?
- AG: Alas, no.
- GT: But, at least, it is a variety, right?
- AG: It depends on what you mean by a *variety*. For instance it can fail to be *irreducible*. Most people (meaning, *most algebraic geometers*) require varieties V to be irreducible, meaning that V cannot be written as a finite union of proper algebraic subsets.
- For instance, the algebraic subset $\{(x, y) : xy = 0\} \subset \mathbf{C}^2$ is reducible. The subset $\{x : x^2 - 1 = 0\} \subset \mathbf{C}$ is also reducible.
- **Convention.** In what follows, an algebraic variety will be always assumed to mean an affine algebraic set, considered up to an isomorphism, no irreducibility assumption will be made.

Example of a reducible character variety

- $\pi = \mathbb{Z}/5$, $X(\pi/G)$ consists of 3 points, represented by ρ_1, ρ_2, ρ_3,
- $$\rho_1 : 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
- $$\rho_2 : 1 \mapsto \begin{bmatrix} e^{2\pi/5i} & 0 \\ 0 & e^{-2\pi/5i} \end{bmatrix},$$
- $$\rho_3 : 1 \mapsto \begin{bmatrix} e^{4\pi/5i} & 0 \\ 0 & e^{-4\pi/5i} \end{bmatrix}.$$

Variety or a scheme?

- GT: Oh, I see, like some of us automatically assume that a *manifold* means *a connected manifold*. But I do not care about this reducibility issue, let us just call this quotient a character variety, because this is the name Culler and Shalen came up with.
- AG: Wait! There is also the character scheme.
- GT: What's that and why do I care? Is it just another mountain to climb?
- AG: Because it's there!
- Me (interrupting the dialog for good): Not only, as we will see, GT will indeed care about this scheme business, once we are done with the background.
- In fact, the "right notion" is not a scheme, but a stack, but let us not go there.

Digression: Everything as a scheme

- Affine algebraic schemes denoted $\mathbb{X}$ or $\mathfrak{X}$ (over **C**) are certain natural functors from (commutative) **C**-algebras to affine algebraic sets $A \mapsto X = \mathbb{X}(A)$, A-points of $\mathbb{X}$.
- For us, schemes will be equivalent to their coordinate rings

$$R = \mathbf{C}[x_1, ..., x_N]/(f_1, ..., f_m)$$

- However, instead of only looking only at complex solutions of the system of equations $f_i(x) = 0$ in N variables,
- we will also consider A-solutions for various algebras A,
- subsets of A^n satisfying the system of equations $f_i(a) = 0$.
- **Example 1.** A group-scheme $\mathbb{G}$, e.g. $\mathbb{SL}_2$,

$$\mathbb{SL}_2 : \mathbf{C} \mapsto \mathbb{SL}_2(\mathbf{C}) = SL(2, \mathbf{C}).$$

Digression: Everything as a scheme

- **Example 2.** The scheme $\{x^k = 0\}$. $R \cong \mathbf{C}[x]/x^k$.
- If we only look at the complex solutions, the only solution we get is $x = 0$.
- But the ring $\mathbf{C}[x]/x^k$ is clearly not isomorphic to the coordinate ring $\mathbf{C} = \mathbf{C}[x]/(x)$ of the scheme $\{x = 0\}$.
- Consider the (commutative) algebra of dual numbers

$$\mathbf{C}[\epsilon] \cong \mathbf{C}[x]/(x^2),$$

where $\epsilon \leftrightarrow x$; $\epsilon^2 = 0$.
- This algebra has zero divisors and, as we will see, it is a good thing.

Digression. Example: Zariski tangent bundle

- Take an affine algebraic scheme $\mathfrak{X}$ given by the ideal $I = (f_1, ..., f_m)$; $L_1, ..., L_m$ are the linear parts of the polynomials $f_1, ..., f_m$.
- For the algebra $A = \mathbf{C}[\epsilon]$ of dual numbers, we consider the set of A-points of $\mathfrak{X}$.
- The result, TX, is the Zariski tangent bundle of the complex variety X, the set of complex points of $\mathfrak{X}$.
- The projection $\xi : TX \to X$ is induced by the homomorphism $D : a + b\epsilon \mapsto a$, where $a, b \in \mathbf{C}$.
- Verification at a point, say, at $\mathbf{0} \in X$:
- The equation $f_i(\mathbf{z}) = 0$, $z = \mathbf{a} + \mathbf{b}\epsilon \in A, \xi(\mathbf{z}) = \mathbf{0}$, amounts to $L_i(\mathbf{b}) = 0$, since $\epsilon^n = 0, n \geq 2$
- and all nonlinear terms drop out.
- Thus, $\xi^{-1}(\mathbf{0}) \cong \{\mathbf{b} \in \mathbf{C}^n : L_i(\mathbf{b}) = 0\} = T_{\mathbf{0}}X$.

More examples

- **Example 3.** The representation scheme

$$\mathfrak{R}ep(\pi, \mathbb{G}) = Hom(\pi, \mathbb{G}) : A \mapsto Hom(\pi, \mathbb{G}(A)).$$

- Thus, complex points of this scheme are representations from π to the complex Lie group $G = \mathbb{G}(\mathbf{C})$.
- **Example 4.** The character scheme

$$\mathfrak{X}(\pi, \mathbb{G}) = \mathfrak{R}ep(\pi, \mathbb{G}) // \mathbb{G}.$$

- The complex points are the elements of the character variety $X(\pi, G)$.
- One would like to have a similar statement over the real numbers (say, for representations to $SL(2, \mathbf{R})$ or $SU(2)$), but it does not work as cleanly.
- One problem is that the quotient of a real algebraic set by a compact group action, in general, is not an algebraic variety, only a semialgebraic variety.

Coordinate rings of representation schemes

- Suppose that G is an algebraic subgroup of $SL(n, \mathbf{C})$; for concreteness, I consider $G = SL(2, \mathbf{C})$.
- Pick a generating set $g_1, \ldots, g_k$ of π and a finite set of relators $r_1, \ldots, r_m$.
- Each representation $\rho : \pi \to G$ determines a point

$$(\rho(g_1), \ldots, \rho(g_k)) \in G^k$$

- which satisfies the set of equations (coming from the relators)

$$r_i((\rho(g_1), \ldots, \rho(g_k))) - I = 0,$$

  where $I \in G$ is the identity matrix.
- Thus, we have $N = 4k$ variables x_i (4 matrix entries for each direct factor of G^k, $G = SL(2, \mathbf{C})$)
- and $M = k + 4m$ equations which I write as $f_j(x) = 0$, $j = 1, ..., M$ (this includes k equations det $= 1$, one per each generator).
- This defines the coordinate ring $R = \mathbf{C}[x_1, ..., x_N]/(f_1, ..., f_M)$ of our representation scheme (not variety!).

More examples

- **Example 4.** Let $\pi \cong \mathbb{Z} = \langle 1 \rangle$; then we have the $SU(2)$-invariant trace function

$$\tau : \rho \mapsto tr(\rho(1)) \in \mathbf{R}$$

- which descends a homeomorphism

$$Hom(\pi, SU(2))/SU(2) \to [-2, 2].$$

- However, each real-algebraic set is homeomorphic to a simplicial complex where links of all vertices have even Euler characteristic.
- Therefore, $[-2, 2]$ is not homeomorphic to a real algebraic set.
- An answer to this puzzle is that by looking at real points of the character scheme $\mathfrak{X}(\pi, SL_2)$, one recovers equivalence classes of representations to $SL(2, \mathbf{R})$ well as to $SU(2)$.

Infinitesimal theory of representation schemes: Tangent spaces and cohomology

- Let $\rho : \pi \to G$, where G is a Lie group.
- Then we have the structure of an π-module on $\mathfrak{g}$, the Lie algebra of G, via composition of ρ and the adjoint representation Ad of G.
- This leads to $Z^1(\pi, Ad\rho)$ and its quotient

$$H^1(\pi, Ad\rho) = Z^1(\pi, Ad\rho)/B^1(\pi, Ad\rho),$$

- the group cohomology with coefficients in the π-module $\mathfrak{g}$.

Digression: DeRham cohomology with coefficients in a flat vector bundle

- There are many ways to define group cohomology with coefficients in a vector space V which is an $\mathbf{F}\pi$-module, $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$.
- For instance, if $\pi = \pi_1(M)$, M is a smooth aspherical manifold, we can proceed as follows.
- Define a flat vector bundle $E_\rho = E \to M$ associated with the action of π on the vector space V:
- $E = (\tilde{M} \times V)/\pi$, $\tilde{M} \to M$ is the universal cover.
- Now consider the deRham complex $\Omega^\bullet(M, E)$, of differential forms on M with coefficients in E.
- Forms locally are $f(x)\omega(x), x \in M$, $f(x) \in V$, $\omega \in \Omega^\bullet(M)$.
- Then $H^*(\pi, V) \cong H^*_{deRham}(M, E)$.

Digression: DeRham cohomology with coefficients in a flat vector bundle

- Among other things, we get the cup-product

$$\cup : H^p(\pi, V) \otimes H^q(\pi, V) \to H^{p+q}(\pi, V \otimes V),$$

- $\xi \cup \eta = \xi \wedge \eta$.
- How to get rid of the tensor product $V \otimes V$:
- When V is equipped with an invariant bilinear form b, e.g. the Killing form on the Lie algebra, we define the pairing
- $\langle \xi, \eta \rangle = b(\xi \wedge \eta) \in H^{p+q}(M, \mathbf{F})$, e.g.

$$b(f(x)dx_i \wedge g(x)dx_j) = b(f(x), g(x))dx_i \wedge dx_j.$$

- We can also use the Lie bracket if $V = \mathfrak{g}$:
- $[\xi, \eta] = [\xi \wedge \eta] \in H^{p+q}(M, \mathfrak{g})$, e.g.

$$[f(x)dx_i \wedge g(x)dx_j] = [f(x), g(x)]dx_i \wedge dx_j.$$

Digression: DeRham cohomology with coefficients in a flat vector bundle

- **Example.** Suppose that M is a closed oriented surface, V is a vector space over a field $\mathbf{F}$.
- Then for
$$\xi, \eta \in H^1(M, E)$$
-
$$\omega(\xi, \eta) = b(\xi \wedge \eta) \in H^2(M, \mathbf{F}) \cong \mathbf{F},$$
- defines an $\mathbf{F}$-valued symplectic form on $H^1(M, E)$.
- In fact, ω is a symplectic form on the smooth part of the character variety $X(\pi_1(M), G)$ (Goldman, 1985).
- We will see relevance of this form for 3-manifolds later on.

Back to character varieties

- **Theorem.** (A. Weil, 1964) If $G = \mathbb{G}(\mathbf{R})$ is an algebraic Lie group then for $\rho : \pi \to G$,

$$T_\rho \mathfrak{R}ep(\pi, \mathbb{G}) \cong Z^1(\pi, Ad\rho).$$

- In fact, the first clean proof I could find is in Raghunathan's book, 1972.
- (A. Sikora, 2012). Suppose that ρ is a smooth point and $\rho : \pi \to G$ is completely reducible. Then:

$$T_\rho \mathfrak{X}(\pi, \mathbb{G}) \cong T_0 \left(H^1(\pi, Ad\rho) // C_G(\rho(\pi)) \right).$$

- If the G-centralizer $C_G(\rho(\pi))$ of $\rho(\pi)$ equals the center of G (which often is the case), then

$$T_\rho \mathfrak{X}(\pi, \mathbb{G}) \cong H^1(\pi, Ad\rho).$$

- **Definition.** ρ is infinitesimally rigid if $H^1(\pi, Ad\rho) = 0$.
- Such $[\rho]$ is necessarily an isolated point of the variety $X(\pi, G)$.

What does “smoothness” mean?

- We now return to the question on when $\mathfrak{R}ep(\pi, \mathbb{G})$ and $\mathfrak{X}(\pi, \mathbb{G})$ are smooth at ρ and $[\rho]$.
- Smoothness will mean that the corresponding variety is a manifold at ρ, resp. $[\rho]$ and its Zariski tangent space (computed via H^1 as above) is isomorphic to the (set of complex points) of the Zariski tangent space of the respective scheme.
- For instance, if ρ is infinitesimally rigid, then both varieties are smooth at ρ, resp. $[\rho]$.
- Warning: The scheme $\{x^2 = 0\}$ is not smooth at $x = 0$, even though, the corresponding variety is.

A smoothness condition

- Smoothness depends (in part) on the 2nd cohomology of π.
- **Theorem.** (Kodaira–Spencer–Goldman–Millson) 1. If $H^2(\pi, Ad\rho) = 0$ then $\mathfrak{Rep}(\pi, \mathbb{G})$ is smooth at ρ.
- 2. If the "bracketed cup-product"

$$[\cdot, \cdot] : H^1(\pi, Ad\rho) \otimes H^1(\pi, Ad\rho) \to H^2(\pi, Ad\rho)$$

  is not identically zero, then $\mathfrak{Rep}(\pi, \mathbb{G})$ is not smooth at ρ.
- Again, if the centralizer of $\rho(\pi)$ equals the center of G and $H^2 = 0$, then one also gets smoothness of the character scheme $\mathfrak{X}(\pi, \mathbb{G})$ at $[\rho]$.
- Thus, the character variety $X(\pi, \mathbb{G})$ is a smooth manifold near $[\rho]$ whose dimension equals the dimension of $H^1(\pi, Ad\rho)$.
- **Note.** In addition to the bracketed cup-product there are "higher smoothness obstructions" coming from the higher Massey products, cf. [GM1987].

2-dimensional case: A. Weil's theorem

- At last, we now specialize to the case of low-dimensional manifolds. I will start with the case of surfaces and surface-orbifolds. These will be relevant when dealing with 3-dimensional Seifert manifolds.
- Let S be a connected compact oriented surface of genus p with q boundary components α_i; $\pi = \pi_1(S)$.
- Let G be an algebraic Lie group.
- Fix a set c of q conjugacy classes c_i in G, $\rho(\alpha_i) \in c_i$, and consider the relative representation scheme

$$\mathfrak{R}ep_c(\pi, \mathbb{G}) = \{\rho \in \mathfrak{R}ep(\pi, \mathbb{G}) : \rho(\alpha_i) \in c_i\}$$

- its $\mathbb{G}$-quotient, the relative character scheme $\mathfrak{X}_c(\pi, \mathbb{G})$.
- Consider the restriction map

$$res : H^1(S, E) \to H^1(\partial S, E)$$

- whose kernel is denoted $H^1_!(S, E)$.

2-dimensional case: A. Weil's theorem

- **Theorem (A. Weil, 1964).**

$$\dim H^1_!(\pi, Ad\rho) = (2p-2)\dim(G) + \sum_{i=1}^{q} \xi_i + \zeta + \zeta^*.$$

- Moreover, if $\zeta^* = 0$ then $\mathfrak{Rep}_c(\pi, \mathbb{G})$ is smooth at ρ.
- Here ζ, ζ^* are the dimensions of $\rho(\pi)$-invariants in $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Thus, ζ is the dimension of the centralizer of $\rho(\pi)$ in G.
- For each i, ξ_i is the dimension of c_i; it equals $\dim(G)$ minus the dimension of the centralizer of $\rho(\alpha_i)$ in G.
- See [P] for a generalization of this formula in the case of non-orientable surfaces.

Weil's theorem: Special cases

- **Note.** Special cases of this theorem were rediscovered many times since Weil, but never the theorem itself in full generality.
- **Special case 1:** $\mathfrak{g} \cong \mathfrak{g}^*$, $q = 0$ (i.e. S has no boundary), then
- $\zeta = \zeta^* = \dim H^0(\pi, Ad\rho) = \dim H^2(\pi, Ad\rho)$ (Poincaré duality) and
- Weil's formula reads

$$\dim H^1(\pi, Ad\rho) = \dim(G)(2p-2) + 2\dim H^0(\pi, Ad\rho),$$

- equivalently, for $d = \dim(G) = \dim(\mathfrak{g})$,

$$d(2-2p) = \dim H^0(\pi, Ad\rho) - \dim H^1(\pi, Ad\rho) + \dim H^2(\pi, Ad\rho),$$

- which amounts to the fact that $\chi(M, E) = d \cdot \chi(M)$.

2-dimensional orbifold case

- **Special case 2:** $\mathcal{O}$ is a closed connected oriented 2-dimensional orbifold without boundary, $M = \mathcal{O} \setminus \Sigma_{\mathcal{O}}$, is the nonsingular set of the orbifold; $\Gamma = \pi_1^{orb}(\mathcal{O})$
- Then Weyl's formula implies:

$$\dim H^1(\Gamma, Ad\rho) = (2p-2)\dim(G) + \sum_{i=1}^{q} \xi_i + \zeta + \zeta^*.$$

- If, furthermore, $\zeta = \zeta^* = 0$ ($\rho(\Gamma)$ has discrete centralizer), then $\zeta^* = \dim H^2(\Gamma, Ad\rho) = 0$
- and we obtain another proof of smoothness of $\mathfrak{R}ep(\Gamma, G)$ at ρ.
- Note that the representation variety in this situation is no longer relative, but the ordinary one,
- this comes from the fact that representations of finite groups Φ into G are infinitesimally rigid:

$$\dim H^1(\Phi, Ad\rho) = 0.$$

Relevance to character varieties of 3-manifold groups

- Let M be a closed aspherical 3-dimensional Seifert manifold with oriented base-orbifold $\mathcal{O}$,
-
$$1 \to \mathbb{Z} \to \pi = \pi_1(M) \xrightarrow{p} \Gamma = \pi_1^{orb}(\mathcal{O}) \to 1.$$
- Consider, say, $\mathfrak{R}ep(\pi, \mathbb{G})$, $G = PSL(2, \mathbf{C})$, and $\rho : \pi \to G$
- whose image is nonabelian. Then $C_G(\rho(\pi)) = 1$ and the pull-back map
$$\mathfrak{X}(\Gamma, \mathbb{G}) \to \mathfrak{X}(\pi, \mathbb{G}), \varphi \mapsto \varphi \circ p,$$
- is a local isomorphism at $[\rho]$.
- **Corollary.** $\mathfrak{X}(\pi, \mathbb{G})$ is smooth at $[\rho]$.
- **Proof.** $\zeta = \zeta^* = 0$, hence, $\mathfrak{X}(\Gamma, \mathbb{G})$ is smooth at $[\varphi]$, $\rho = \varphi \circ p$. □

Dimension count

- **Theorem.** (W. Thurston; M. Culler, P.Shalen) Let M be a compact oriented 3-dimensional manifold with boundary.
- G is a complex semisimple Lie group.
- Then dimension of each irreducible component of $X(M, G)$ is at least $-\dim(G)\chi(M)$.

Character varieties of 3-manifold groups: Lagrangian structure

- Let M be a compact oriented 3-dimensional manifold with boundary,
- G is a reductive group over the field $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$, its Lie algebra is equipped with nondegenerate bilinear form

$$tr : \mathfrak{g} \otimes \mathfrak{g} \to \mathbf{F},$$

- For $E = E_\rho$, on $H^1(\partial M, E)$ we also have the $\mathbf{F}$-symplectic form

$$\omega(\xi, \eta) = tr(\xi \cup \eta) \in \mathbf{F}.$$

- **Warning:** The boundary could be disconnected.
- **Theorem.** (A. Sikora, 2012) The image of the restriction map

$$res : H^1(M, E) \to H^1(\partial M, E)$$

is a Lagrangian subspace; in particular, it is half-dimensional.

Character varieties of 3-manifold groups: Lagrangian structure

- On the level of representation and character schemes we have the restriction morphism

$$res : \mathfrak{X}(\pi_1(M), \mathbb{G}) \to \mathfrak{X}(\pi_1(\partial M), \mathbb{G}),$$

- $\partial M = \bigcup_{i=1}^{n} \Sigma_i$ is the disjoint union of closed surfaces, and we set
-
$$\mathfrak{R}ep(\pi_1(\partial M), \mathbb{G}) := \prod_{i=1}^{n} \mathfrak{R}ep(\pi_1(\Sigma_i), \mathbb{G})$$
- and, similarly, for the character schemes.
- Then $res(\rho) = (\varphi_1, ..., \varphi_n)$,
- $\varphi_i = \rho|\pi_1(\Sigma_i)$. (Ignoring the base-points issue.)

Character varieties of 3-manifold groups: Lagrangian structure

- Then we can interpret the cohomological result as saying that the restriction map sends $\mathfrak{X}(\pi_1(M), \mathbb{G}) \to \mathfrak{X}(\pi_1(\partial M), \mathbb{G})$ to a formally lagrangian subscheme.
- The trouble with this interpretation is that none of the schemes here is, a prori, smooth, and the 1st cohomology, in general, is not even isomorphic to the Zariski tangent space.
- Assume, for a moment, that the *character varieties* $X(\pi_1(M), G), X(\pi_1(\partial M), G)$ are smooth at ρ and that the image of *res* is also smooth at ρ (all true generically).
- One would like to conclude that, at least in this setting,
- $Y := res(X(\pi_1(M), G))$ is a Lagrangian submanifold of $X(\pi_1(\partial M), G)$ at $[\rho]$.
- What we know, however, only is that, in this situation, the symplectic form ω vanishes on $T_{[\rho]}(Y)$, but not that $T_{[\rho]}(Y)$ is half-dimensional!

Character varieties of 3-manifold groups: Lagrangian structure

- The reason that we do not know this is that H^1 only controls the Zariski tangent space on the scheme-theoretic level, not on the level of the variety.
- **Problem.** Let M be a hyperbolic knot complement. Is it true that each irreducible component of

$$Y = res(X(\pi_1(M), SL(2, \mathbf{C})) \subset X(\mathbb{Z}^2, SL(2, \mathbf{C}))$$

  has (complex) dimension 1? (Probably false.)
- Note that dimension 2 is impossible since ω is a nondegenerate form on $X(\mathbb{Z}^2, SL(2, \mathbf{C}))$.
- The trouble is that, for all what we know, images of some components of $X(\pi_1(M), SL(2, \mathbf{C}))$ can be non-reduced, 0-dimensional, but with 1-dimensional Zariski tangent space at each point (on the scheme-level), cf. [PP].
- We will come back to this when discussing the A-polynomial of knots.

Boundary restriction map

- Note that, in general, there is no reason for the restriction map
$$res : H^1(M, E) \to H^1(\partial M, E)$$
to be 1-1, or of constant rank, which creates extra problems.
- Suppose that $int(M)$ admits a complete hyperbolic structure (possibly of infinite volume) and
- $\rho_0 : \pi_1(M) \hookrightarrow PSL(2, \mathbf{C})$ is the corresponding "uniformizing" representation.
- Then ρ_0 lifts (in general, nonuniquely!) to a representation $\rho : \pi \to SL(2, \mathbf{C})$ (M. Culler, 1986).
- On the smoothness and cohomology side, it does not matter if we consider ρ_0 or ρ.
- The next theorem is a special case of a more general infinitesimal rigidity result (Calabi, Weil, Matsushima, Murakami, Raghunathan ...)

Boundary restriction map

- **Theorem.** The restriction map

$$res : H^1(M, E) \to H^1(\partial M, E)$$

is 1-1.

- Moreover:
- **Theorem (Kapovich, 1991–2000).** Assume that $\pi = \pi_1(M)$ is nonabelian. Then $\mathfrak{X}(\pi, SL(2, \mathbf{C}))$ is smooth at $[\rho]$, the boundary restriction map is an immersion and its image in $X(\partial M, SL(2, \mathbf{C}))$ is (locally) Lagrangian.
- A very useful special case is when $int(M)$ is of finite volume, with n cusps.
- **Corollary.** Then $X(\partial M, SL(2, \mathbf{C}))$ is complex $2n$-dimensional and contains an irreducible n-dimensional component $X_o \subset X(M, SL(2, \mathbf{C}))$ through $[\rho]$, which is (generically) Lagrangian. (W. Thurston, 1970s.)

Rigidity theorems

- The injectivity theorem from the previous page extends to higer-dimensional representations induced by irreducible representations

$$r_d : SL(2, \mathbf{C}) \to SL(d, \mathbf{C})$$

- Let $\rho_d : \pi \hookrightarrow SL(2, \mathbf{C})$ be the composition of r_d with the uniformizing representation $\rho : \pi \hookrightarrow SL(2, \mathbf{C})$.
- Let $E_d \to M$ be the corresponding flat $sl(d, \mathbf{C})$ vector bundle.
- **Theorem (Matsushima-Murakami, Raghunathan).** The restriction map

$$res : H^1(M, E_d) \to H^1(\partial M, E_d)$$

is 1-1.
- **Corollary.** If M has no toral boundary components then $\mathfrak{X}(\pi, SL(d, \mathbf{C}))$ is smooth at $[\rho_d]$, of complex dimension

$$-(d^2 - 1)\chi(M).$$

- In particular, if M is closed then $[\rho_d]$ is infinitesimally rigid.

Smoothness problem

- Suppose that M is closed, but instead of embedding into complex groups, we take

$$SO(3,1) = Isom(\mathbb{H}^3) \hookrightarrow SO(4,1) = Isom(\mathbb{H}^4)$$

- and consider the character scheme

$$\mathfrak{X}(\pi, SO(4,1))$$

  at the equivalence class of the uniformizing representation, $[\rho]$.
- At this point, there is no need to assume orientability of M.
- By Thurston's holonomy theorem, this deformation problem corresponds to the deformation problem for the hyperbolic structure on M among flat conformal structures on M.
- **Question.** Is it true that $[\rho]$ is a smooth point?
- **Theorem.** Smoothness fails if, instead, we deform hyperbolic n-manifold groups in $SO(n+1,1)$, $n \geq 4$ (Johnson, Millson, 1984).

Smoothness problem

- **Theorem (Kapovich, Millson, 1996, [KM1996])**. The bracketed cup-product

$$[\cdot,\cdot] : H^1(M,E) \times H^1(M,E) \to H^2(M,E)$$

is identically zero in this setting.
- Thus, the first obstruction to smoothness is zero.

Smoothness problem

- If one allows not only hyperbolic manifolds but also hyperbolic orbifolds, then in one case smoothness does hold:
- **Theorem.** (Kapovich, 1994) Let $\mathcal{O}$ be a closed 3-dimensional hyperbolic reflection orbifold, $\pi = \pi_1^{orb}(\mathcal{O})$, which has n boundary mirrors.
- Then $\mathfrak{X}(\pi, SO(4,1))$ is smooth at $[\rho]$ and has dimension $n - 4$ at that point.
- Back to manifolds, in addition to smoothness, one can ask if $[\rho]$ is an isolated point.
- **Theorem (Johnson–Millson, 1984).** Suppose that M contains a properly embedded totally-geodesic hypersurface. Then $[\rho]$ is not isolated. (Same in higher dimensions.)
- On the other hand, there are examples which are infinitesimally rigid.

Local rigidity: $SO(4,1)$ setting

- **Theorem.** (Kapovich, 1994; Porti–Francaviglia, 2008, stronger form.) Suppose that K is a 2-bridge knot in S^3.
- Then for all but finitely many manifolds M obtained via Dehn surgery on K,
- $[\rho_M]$ is an isolated point of $X(\pi, SO(4,1))$,
- $$\rho_M : \pi_1(M) \hookrightarrow SO(3,1).$$
- Moreover, the representations ρ_M are infinitesimally rigid.

Local rigidity: $SL(4, \mathbf{R})$ setting

- Instead of deforming the uniformizing representation composed with $SO(3,1) \hookrightarrow SO(4,1)$,
- one can look at the deformation problem for the composition

$$\rho : \pi \hookrightarrow SO(3,1) \hookrightarrow SL(4, \mathbf{R}).$$

- For geometric structures, this deformation problem corresponds to deformations of the hyperbolic structure on M among all real-projective structures on M.
- **Theorem (Heusener, Porti, 2011).** There are infinitely many closed hyperbolic 3-manifolds M such that $\rho : \pi_1(M) \hookrightarrow SL(4, \mathbf{R})$ is infinitesimally rigid.
- On the other hand, for closed hyperbolic manifolds M containing properly embedded totally geodesic hypersurfaces, $[\rho]$ is not an isolated point in $X(\pi_1(M), SL(4, \mathbf{R}))$ (Johnson, Millson, 1984).

How common is rigidity?

- At this point, it is very far from clear what topological invariants of hyperbolic 3-manifolds are responsible for their rigidity and nonrigidity. For instance, there are (K. Scannell, 2002) hyperbolic 3-manifolds containing quasifuchsian incompressible surfaces which are locally rigid (in the $SO(4,1)$ setting).
- Cooper, Long and Thistlethwaite (2006) looked at the list of 4500 hyperbolic manifolds from the Hodgson–Weeks census, which have rank 2 fundamental group.
- They proved that for at most 61 of these manifolds the point $[\rho]$ is not isolated in $X(\pi_1(M), SL(4, \mathbf{R}))$.
- Furthermore, they found examples which are not infinitesimally rigid, yet, are isolated in $X(\pi_1(M), SL(4, \mathbf{R}))$.

Main theorem: Universality/Murphy's Law

- **Theorem (Kapovich, Millson, 2013).** Let X be an affine scheme of finite type over $\mathbf{Q}$ and $x_0 \in X$ a rational point. Then there exist:
- A natural number m.
- A closed 3-dimensional manifold M with the fundamental group π.
- A representation $\rho_0 : \pi \to SU(2)$, with dense image.
- An isomorphism of germs

$$(\mathfrak{R}ep(\pi, SL(2)), \rho_0) \to (X \times \mathbb{A}^m, x_0 \times 0)$$

  where $\mathbb{A}^m$ is the affine m-space.
- In other words, we can realize any singularity (over $\mathbf{Q}$) as a singularity of a representation scheme at the expense of adding "dummy variables" to the equations.

A universality conjecture

- **Conjecture.** Universality also holds for the following classes of groups Γ:

1. Fundamental groups of closed orientable hyperbolic 3-manifolds.
2. Fundamental groups of hyperbolic knot complements in S^3.

- That is, their character schemes $\mathfrak{X}(\Gamma, SL(2))$ can have "arbitrary" singularities over $\mathbf{Q}$.
- **Theorem (Kapovich, Millson, 1996, [KM1996]).** There are closed hyperbolic 3-manifolds such that $\mathfrak{X}(\pi, SL(2))$ has non-quadratic singularities at some irreducible unitary representations.

An even more irresponsible conjecture

- **Conjecture.** Given any affine scheme X there exists a closed oriented irreducible 3-manifold M such that the character scheme $\mathfrak{X}^{irr}(\pi_1(M), SL(2))$ of irreducible representations, is isomorphic to X.
- Another partial piece of evidence towards this conjecture is a recent work of Kuperberg and Samperton on the complexity (NP hardness) for the problem of existence of nontrivial representations of closed hyperbolic 3-manifold groups to finite simple groups.

Relation to topology

- M is a compact oriented 3-manifold with whose boundary is a single 2-torus.
- For instance, if K is a nontrivial knot in S^3 and M is the complement to its open regular neighborhood $N(K)$.
- It has emerged from the work of Culler and Shalen in 1980s, followed by the one of Culler, Gordon, Luecke and Shalen, and then Cooper, Culler, Gillet, Long and Shalen,
- that the geometry of $X = X(\pi, SL(2, \mathbf{C}))$ sheds some light on topology of the manifold M.
- Let λ, μ be the isotopy classes of loops on the boundary torus T^2 of M, generating $H_1(T^2)$.
- For instance, in the case of a knot complement, we can take λ, μ to be respectively the *longitude* and the *meridian* of K (μ bounds a disk in $N(K)$ and λ is homologically trivial in M).

A-polynomial: The setup.

- We have the boundary restriction map $res : X = X(M, SL(2, \mathbf{C})) \to X(\mathbb{Z}^2, SL(2, \mathbf{C}))$.
- Take $X' \subset X$ to be the union of irreducible components X_i of X such that $res(X_i)$ is 1-dimensional.
- The space $X(\mathbb{Z}^2, SL(2, \mathbf{C}))$ is the quotient of $\mathbf{C}^* \times \mathbf{C}^*$ by the involution
$$(\xi, \eta) \mapsto (\xi^{-1}, \eta^{-1}).$$
- This quotient comes from the fact that a diagonalizable representation does not have a canonical diagonalization; for a representation $\rho : \langle\lambda\rangle \oplus \langle\mu\rangle$ we have that the eigenvalues of $\rho(\lambda)$ are $\{\xi, \xi^{-1}\}$ and for $\rho(\mu)$ they are $\{\eta, \eta^{-1}\}$.
- This ambiguity of which eigenvalue to take is ennoying, but not more than this.
- Now, lift $res(X')$ to $\mathbf{C}^* \times \mathbf{C}^* \subset \mathbf{C}^2$. The lift is a (typically not irreducible) complex affine curve, whose closure is denoted C.

A-polynomial: The definition.

- We now have an affine algebraic curve $C \subset \mathbf{C}^2$ derived from the character variety. (Note that the scheme–theoretic structure is ignored here.)
- Let $I_k \subset \mathbf{C}[x, y]$ be the ideal of polynomials vanishing on the irreducible component $C_k \subset C$; this ideal is principal, generated by a polynomial $p_k(x, y)$, unique up to multiplication by a nonzero complex number.
- Now, take $p = p_1 \dots p_n$, n is the number of irreducible components of C. Note that p depends on M as well as on the choice of the basis in $H_1(T^2)$.
- We assume now that M is a knot complement, in which case the basis (consistent with the orientation of the torus) is uniquely defined, except we can simultaneously invert the loops λ, μ.
- Since this inversion is already taken care of by the map

$$\mathbf{C}^* \times \mathbf{C}^* \to X(\mathbb{Z}^2, SL(2, \mathbf{C})),$$

- the p (still definited up to a scalar) now depends only on K.

A-polynomial: The definition.

- The curve C contains a distinguished component $\{x = 1\}$, coming from representations $\pi \to SL(2, \mathbf{C})$ with cyclic image,
- $$\rho : \lambda \mapsto 1 \in SL(2, \mathbf{C}), \xi(\rho) = 1.$$
- Thus, p is divisible by $x - 1$.
- **Definition.**
$$A_K(x, y) = \frac{1}{x-1} p(x, y)$$
is the A-polynomial of the knot K.
- One can then rescale A to obtain a polynomial with integer coefficients; as the result, A_K is defined up to ± 1.

A-polynomial: Nontriviality.

- If $int(M)$ is hyperbolic, then X contains a distinguished irreducible component X_o containing (the equivalence class of) the uniformizing representation.
- In this case, it is clear that A_K is nonconstant.
- Nontriviality of A_K for general unknots was an open problem until the work of Kronheimer and Mrowka (2004), who proved existence of noncyclic $SU(2)$-representations of fundamental groups of nontrivial knot complements (after infinitely many Dehn surgeries).
- **Theorem (Dunfield, Garoufalidis, 2004; Boyer–Zhang, 2005).** The A-polynomial A_K of each nontrivial knot K is nontrivial.
- **Theorem (Boden, 2014).** For each nontrivial knot K, the polynomial A_K has nonzero degree in the y-variable.

A-polynomial: Elementary properties.

1 The curve C cannot contain either one of the coordinate lines $x = 0$, $y = 0$.

2 Hence, the polynomial A has nonzero constant term.

3 C is invariant under the involution $(x, y) \mapsto (1/x, 1/y)$.

4 **The Newton polygon** $Newt(A) \subset \mathbf{R}^2$ of

$$A(x, y) = 1 + \sum_{i,j=1}^{i_{max}, j_{max}} a_{ij} x^i y^j$$

is the convex hull of the set (i, j) such that $a_{ij} \neq 0$.

5 Parts 2, 3 imply that $(0, 0) \in Newt(A)$ and $Newt(A)$ is invariant under the involution

$$(i, j) \mapsto (i_{max} - i, j_{max} - j)$$

6 Since $j_{max} > 0$, the polygon is not contained in the x-axis.

7 By Part 4, it cannot be contained in any horizontal line.

Character varieties and boundary slopes.

- **Definition.** A boundary loop of a 3-manifold M (with a single toral boundary component) is a simple homotopically nontrivial loop τ on ∂M, such that τ bounds an incompressible surface $(S, \partial S)$ in $(M, \partial M)$.
- The slope of a boundary loop τ is the ratio $b/a \in \mathbf{Q} \cup \{\infty\}$,
- where $\tau = \lambda^a \mu^b$.
- A boundary slope of M is a ratio $b/a \in \mathbf{Q} \cup \{\infty\}$ which is the boundary slope of a boundary loop.
- It turns out, that the *Newton polygon* of the A-polynomial contains some nontrivial information about boundary slopes of M.
- **Theorem (Cooper, Culler, Gillet, Long and Shalen, 1994).** The slopes of boundary edges of $Newt(A_K)$ are all boundary slopes of the knot K (or, more precisely, of its complement).

Separating incompressible surfaces.

- Below is another application of the character variety $X(\pi, SL(2, \mathbf{C}))$ to the topology of M.
- For dimension reasons, the map $H_1(\partial M, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is not injective.
- From this, one concludes by appealing to the Loop Theorem, that there exists a *nonseparating incompressible surface* with nonempty boundary $(S, \partial S) \subset (M, \partial M)$, this surface bounds a nonseparating loop in ∂M.
- **Theorem (Weak Neuwirth Conjecture). (Culler, Shalen, 1984)** Each nontrivial knot complement M contains a nonseparating incompressible surface with nonempty boundary.
- **Proof.** The longitude (which has zero slope) is the only boundary loop which can bound a nonseparating surface.
- But $Newt(A)$ is not contained in a horizontal line, hence, it has an edge of nonzero slope.
- Hence, M has a nonzero boundary slope. □

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