# Coordinates for character varieties II
# Shape coordinates for $\mathrm{PGL}(3,\mathbb{C})$

Antonin Guilloux

CURVE Conference

# Goals

- Describe shape coordinates for $\mathrm{PGL}(2, \mathbf{C})$ and $\mathrm{PGL}(3, \mathbf{C})$.

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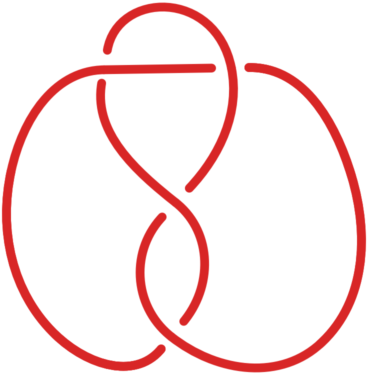
- Describe the holonomy map.
- Some hints about computations.
- A description of the algebraic variety for the 8-knot complement.

# The 8-knot complement



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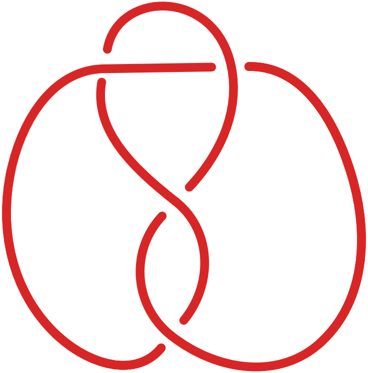

$M_8$ is the 8-knot complement.
Its fundamental group is $\Gamma_8 = \langle\ a, b \mid\ ab^3aBA^2B\ \rangle$.

# Triangulation of $M_8$

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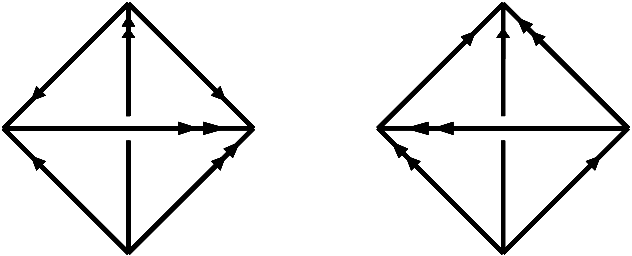

$M_8 \simeq$ is the union of both tetraedra minus vertices. At the end, a complex of 2 tetraedra, with 4 faces, 2 edges, 1 vertex "at infinity".

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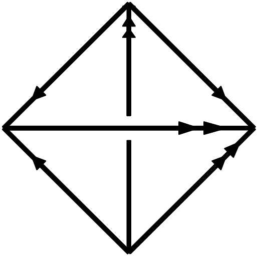
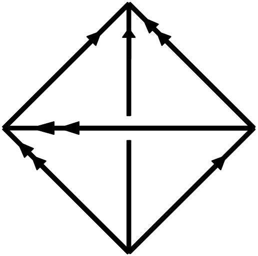


$z$



$w$

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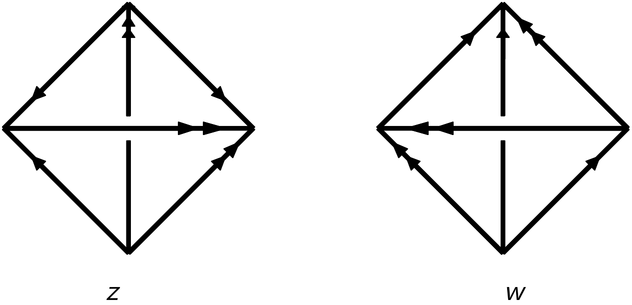


Edge equation :

$$z^2 w^2 \frac{1}{(1-z)(1-w)} = 1.$$

# Parametrize the $\mathrm{SL}(2,\mathbf{C})$-character variety (Thurston)

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## The *deformation variety*

$$\mathrm{Defor}_2(M_8) = \left\{ z, w \in \mathbf{C} \text{ s.t. } z^2 w^2 \frac{1}{(1-z)(1-w)} = 1 \right\}$$

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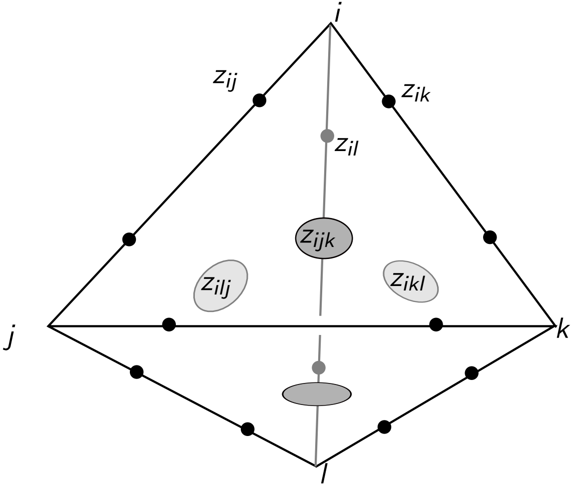

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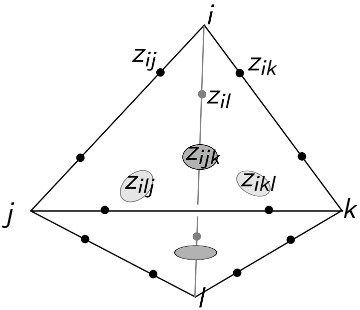
16 parameters for each tetrahedra with internal relations.

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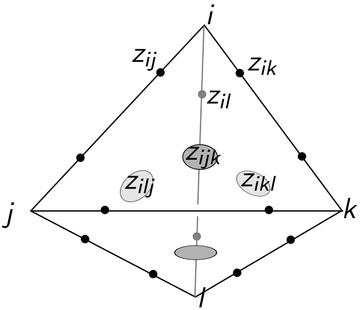


Conditions:

- For each face $z_{ijk} = -z_{il}z_{jl}z_{kl}$

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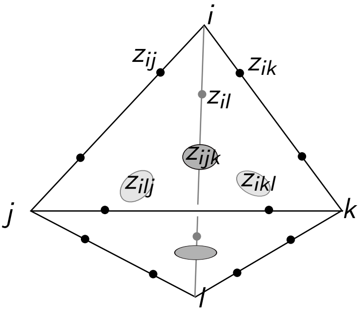


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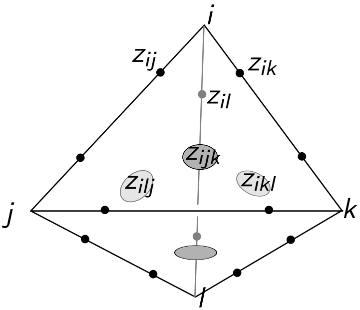


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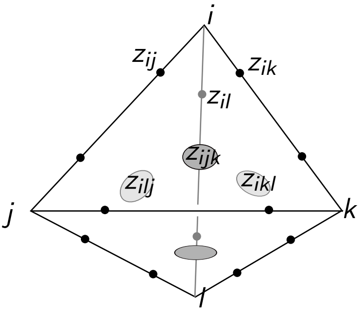


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$z_{ij} \in \mathbf{C} \setminus \{0, 1\}$ and
$z_{ijk} \in \mathbf{C} \setminus \{1\}$.

Remark: A tetrahedron of flags comes from a hyperbolic tetrahedron iff $z_{ij} = z_{ji}$

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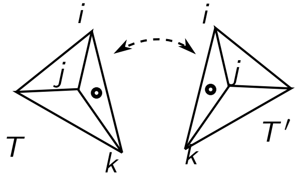


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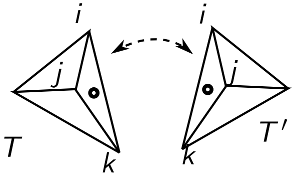


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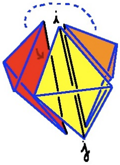
**Edge equations:**



$$z_{ij}(T_1)z_{ij}(T_2)z_{ij}(T_3)\cdots = 1.$$

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- Theoretical input (Falbel-G.) : the interesting components of $\mathrm{Defor}_3(M_8)$ have dimension at least 2.

# Base changes

Through the choice of a normalisation, there is a base of $\mathbf{C}^3$ corresponding to each triple $(F_1, F_2, F_3)$ of (projective) flags.

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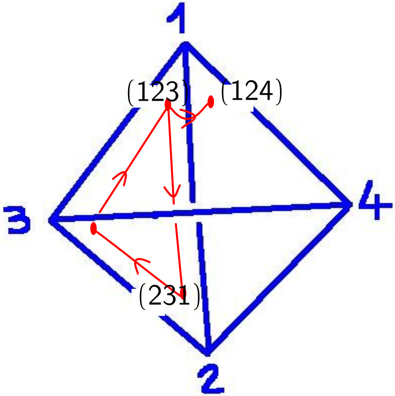

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$$(123) \to (231) = T(z_{123}) = \begin{pmatrix} z_{123} & z_{123}+1 & 1 \\ -z_{123} & -z_{123} & 0 \\ z_{123} & 0 & 0 \end{pmatrix}$$

$$(123) \to (124) = E(z_{12}, z_{21}) = \begin{pmatrix} z_{21}^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_{12} \end{pmatrix}$$

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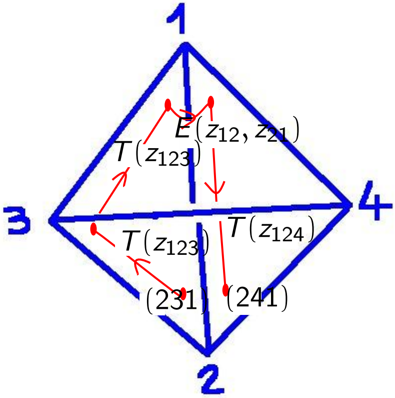
Example: the right turn



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$$(231) \to (241) = R = \begin{pmatrix} z_{12} z_{241} & \star & \star \\ & z_{231} & \star \\ & & \frac{z_{231}}{z_{21}} \end{pmatrix}$$

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Under the face conditions, you can glue the red complex along faces, and under edge conditions the loop around an edge has a trivial holonomy.

## The holonomy map

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The peripheral holonomy is easily computed.

# A theorem

**Theorem (Falbel-G.-Koseleff-Rouillier-Thistlethwaite)**

$\mathrm{Defor}_3(V_8)$ is the union of 3 affine algebraic smooth irreducible variety of dimension 2.

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Very important for applications, e.g. constructing and deforming geometric structures.

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## An exemple

On a component (the geometric one):

$$a \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -u \\ 0 & 1 & \frac{(-1+v)^2(1+v)}{u} \end{bmatrix} \text{ and}$$

$$b \mapsto \begin{bmatrix} 1 & \frac{-u^3(-2+v)+(-1+v)^3(1+v)}{u(-1+v)^2v} & -\frac{u(-2+v)}{-1+v} \\ 0 & v & \frac{-v+v^3}{u} \\ 0 & \frac{u}{v-v^2} & -1 \end{bmatrix}$$